積分技巧-三角函數積分

2021年5月7日 星期五　　下午5:30

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

$\int \sin x \cos^3 x \, dx$

let $u = \cos x$

$du = -\sin x \, dx$

$dx = \frac{1}{-\sin x} du$

$\Rightarrow \int \cancel{\sin x}\, u^3 \cdot \frac{1}{\cancel{-\sin x}} du$

$= -\int u^3 du = -\frac{1}{4} u^4$

$= -\frac{1}{4}\cos^4 x + C$

$\int t \sin^2 t \, dt$

$\Rightarrow \int t \underbrace{\sin^2 t}_{\sin^2 t = \frac{1}{2}(1-\cos(2t))} dt = \frac{1}{2}\int t - t\cos(2t) \, dt$

$= \frac{1}{2}\left(\frac{1}{2}t^2 - \left(\frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t)\right)\right)$

$= \frac{t^2}{4} - \frac{1}{4}t\sin(2t) - \frac{1}{8}\cos(2t) + C$

	D	I
+	t	$\cos(2t)$
−	1	$\frac{1}{2}\sin(2t)$
+	0	$-\frac{1}{4}\cos(2t)$
−		

$\int \cos^2 x \tan^3 x \, dx$

$\Rightarrow \int \cos^2 x \left(\frac{\sin^3 x}{\cos^3 x}\right) dx$

$= \int \frac{\sin^3 x}{\cos x} dx = \int \frac{\sin^2 x \cdot \sin x}{\cos x} dx$

$= \int \frac{(1-\cos^2 x)\sin x}{\cos x} dx$

let $u = \cos x$, $du = -\sin x \, dx$

$dx = -\frac{1}{\sin x} du$

$\Rightarrow -\int \frac{1-u^2}{u} \cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}} du$

$= -\int \frac{1}{u} - u \, du = -\ln|u| + \frac{1}{2}u^2$

$= -\ln|\cos x| + \frac{1}{2}\cos^2 x + C$

$\int \frac{\cos x + \sin(2x)}{\ldots}$

$$\int \frac{\cos x+\sin(2x)}{\sin x}\,dx$$

$\sin(2x) = 2\sin x\cos x$

$$\Rightarrow \int \frac{\cos x}{\sin x} + \frac{\sin(2x)}{\sin x}\,dx$$

let $\sin x = u$
$du = \cos x\,dx$
$dx = \frac{1}{\cos x}du$

$\int \frac{\cancel{\cos x}}{u}\cdot\frac{1}{\cancel{\cos x}}du$
$= \int \frac{1}{u}du$

$$= \int \frac{1}{u}du + \int 2\cos x\,dx$$

$$= \ln|u| + 2\sin x = \ln|\sin x| + 2\sin x + C$$

$$\int \csc x\,dx$$

$$\Rightarrow \int \frac{\csc x\,(\csc x+\cot x)}{\csc x+\cot x}\,dx$$

$$= \int \frac{\csc^2 x + \csc x\cot x}{\csc x+\cot x}\,dx$$

let $u = \csc x + \cot x$

$du = -\csc x\cot x - \csc^2 x\,dx$

$$\Rightarrow \int \frac{-1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\csc x+\cot x| + C$$

$$\int \sec x\,dx$$

$$\Rightarrow \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx$$

let $\sec x + \tan x = u$

$du = \tan x\sec x + \sec^2 x\,dx$

$$\Rightarrow \int \frac{\sec^2 x+\tan x\sec x}{u}\cdot\frac{1}{\sec^2 x+\tan x\sec x}\,du$$

$$= \int \frac{1}{u}\,du = \ln|u| = \ln|\sec x+\tan x| + C$$

$$\int \cos x\,dx$$

$$\Rightarrow \sin x + C$$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$$\int \cos^2 x\,dx$$

$\cos^{2}x = \frac{\quad}{2}$

$\int \cos^2 x \, dx$

$\Rightarrow \int \frac{1}{2}(1+\cos(2x))\,dx$

$= \frac{1}{2}\int 1+\cos(2x)\,dx$

$= \frac{1}{2}\left(x+\frac{1}{2}\sin(2x)\right)$

$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

$\int \cos^3 x \, dx$

$\Rightarrow \int \cos^2 x \cos x \, dx$

$= \int (1-\sin^2 x)\cos x \, dx$

$\cos^2 x = 1-\sin^2 x$

$\Rightarrow$ let $u=\sin x$ $\quad du = \cos x \, dx$

$dx = \frac{1}{\cos x}du$

$\Rightarrow \int 1-u^2 \, du = u - \frac{1}{3}u^3$

$= \sin x - \frac{1}{3}\sin^3 x + C$

$\int \cos^4 x \, dx$

$= \int (\cos^2 x)^2 \, dx$

$= \int \left(\frac{1}{2}(1+\cos(2x))\right)^2 dx$

$= \frac{1}{4}\int 1 + 2\cos(2x) + (\cos 2x)^2 \, dx$

$= \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2}(1+\cos(2\cdot 2x)) \, dx$

$= \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos 4x \, dx$

$\frac{3}{2}$

$= \frac{1}{4}\left(\frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x)\right)$

$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$

$\int \sin \, dx$

$\Rightarrow -\cos x + C$

$\int \sin^2 x \, dx \qquad$ ☆ $\sin^2 x = \frac{1-\cos 2x}{2}$

$\Rightarrow \int \frac{1-\sin 2x}{2} dx$

$= \int \frac{1}{2} - \frac{1}{2}\sin 2x \, dx$

$= \frac{1}{2}x + \frac{1}{4}\cos(2x) + C$

$\int \sin^3 x \, dx$

$\Rightarrow \int \sin x \, (\sin^2 x) \, dx$

$= \int \sin x \, (1 - \cos^2 x) \, dx$

$\Rightarrow$ let $u = \cos x$

$du = -\sin x \, dx$

$dx = -\frac{1}{\sin x} du$

$\Rightarrow \int (1 - u^2) \cancel{\sin x} \cdot \left(-\frac{1}{\cancel{\sin x}} du\right)$

$= \int u^2 - 1 \, du$

$= \frac{1}{3}u^3 - u$

$= \frac{1}{3}\cos^3 x - \cos x + C$

$\int \sin^4 x \, dx$

$\Rightarrow \int (\sin^2 x)^2 \, dx$ ☆ $\cos^2 x = \frac{1 + \cos 2x}{2}$

$= \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx$

$= \frac{1}{4}\int (1 - \cos 2x)^2 \, dx$

$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x \, dx$ 再積分

$= \frac{1}{4}\left[x - \sin 2x + \int \frac{1}{2}(1 + \cos 4x)\, dx\right]$

$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1}{2} + \frac{1}{2}\cos 4x \, dx$

$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{4}\left(\frac{1}{2}x + \frac{1}{8}\sin 4x\right)$

$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x$

$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

$\int \tan x \, dx$

$\Rightarrow \int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

$\Rightarrow$ let $u = \cos x$

$du = -\sin x \, dx$

$$dx = -\frac{1}{\sin x}\,du$$

$$\Rightarrow \int \frac{\sin x}{u} \cdot -\frac{1}{\sin x}\,du$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u|$$

$$= -\ln|\cos x| + C$$

or

$$= \ln|(\cos x)^{-1}| + C$$

$$= \ln|\sec x| + C$$

$$\int \tan^2 x\,dx$$

$$\Rightarrow \int \tan^2 x\,dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x}\,dx$$

$$= \int \frac{1-\cos^2 x}{\cos^2 x}\,dx \qquad \sin^2 x = 1-\cos^2 x$$

$$= \int \frac{1}{\cos^2 x} - 1\,dx$$

$$= \int \sec^2 x - 1\,dx$$

$$= \tan x - x + C$$

$$\int \tan^3 x\,dx$$

$$\Rightarrow \int \tan^3 x\,dx \qquad \begin{aligned}\tan^2 x + 1 &= \sec^2 x \\ \tan^2 x &= \sec^2 x - 1\end{aligned}$$

$$= \int \tan x \cdot (\sec^2 x - 1)\,dx$$

$$= \int \tan x \sec^2 x - \tan x\,dx$$

$$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

let $u = \tan x$

$du = \sec^2 x\,dx$

$dx = \frac{1}{\sec^2}\,du$

上面求過

$\int \tan x\,dx$

$= -\ln|\cos x| + C$

$$\Rightarrow \int u \sec^2 x \cdot \frac{1}{\sec^2 x}\,du + \ln|\cos x|$$

$$= \int u\,du + \ln|\cos x|$$

$$= \frac{1}{2}u^2 + \ln|\cos x|$$

$$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$$