對數函數

2021年3月21日 星期日　　下午11:20

$$e^{\ln x} = x$$

$$\Rightarrow \underbrace{e^{\ln x}}_{x} \frac{d}{dx}[\ln x] = 1$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

☆對數律:

$$\ln x^a = a \ln x$$

$$\ln a + \ln b = \ln(ab)$$

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

$$\log_a N = \frac{\ln N}{\ln a}$$

$$\frac{d}{dx}\left[\ln\left(\frac{1}{x}\right)\right]$$

$$\Rightarrow \ln(x^{-1})$$

$$= \frac{1}{x^{-1}} \cdot -x^{-2}$$

$$= -x \cdot \frac{1}{x^2} = -\frac{1}{x}$$

$$\frac{d}{ds}[\ln(\ln(s))]$$

$$\Rightarrow \frac{1}{\ln(s)}\left(\frac{1}{s}\right)$$

$$\frac{d}{dx}[\ln(e^{-x} + xe^{-x})]$$

$$\Rightarrow \frac{1}{e^{-x} + xe^{-x}}\left(-e^{-x} + e^{-x} - xe^{-x}\right)$$

$$= \frac{-xe^{-x}}{e^{-x} + xe^{-x}} = \frac{-x}{1+x}$$

or $\frac{d}{dx}[\ln(e^{-x}(1+x))]$

$$= \frac{d}{dx}[\underbrace{\ln e^{-x}}_{-x} + \ln(1+x)] = -1 + \frac{1}{1+x}$$

$$= \frac{-x}{1+x}$$

$$\frac{d}{dx}[x^{\ln x}]$$

$\Rightarrow$ let $y = x^{\ln x}$

↳ 同時取 $\ln$

$$\ln y = \ln x^{(\ln x)}$$

$$\ln y = \ln x \ln x = [\ln x]^2$$

$$\frac{1}{y}\frac{d}{dx}[y] = 2(\ln x)\frac{1}{x}$$

$$\frac{d}{dx}y = \frac{2\ln x}{x}\cdot x^{\ln x}$$

$$\frac{d}{dx}\left[\sqrt[3]{\frac{x+1}{x-1}}\right]$$

let $y = \sqrt[3]{\frac{x+1}{x-1}}$ $\qquad (x+1)\left(\frac{1}{x-1}\right)$ $(x-1)^{-1}$

$$\ln y = \ln\left[\left(\frac{x+1}{x-1}\right)^{\frac{1}{3}}\right]$$

$$\ln y = \frac{1}{3}\ln\left(\frac{x+1}{x-1}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\left(\frac{x-1}{x+1}\right)\left[\frac{1}{x-1} - (x+1)(x-1)^{-2}\right]$$

換 x 再乘上去!

$$\frac{d}{dx}[\log_2 x]$$

$\frac{\ln x}{\ln 2}$] 換底公式

$$\Rightarrow \frac{d}{dx}\left[\ln x(\ln 2)^{-1}\right]$$

$$= (\ln 2)^{-1}\frac{1}{x}$$

$e^{\square} = 2$

$$e^{\ln 2} = e^{\log_e 2} = 2$$

$$\frac{d}{dx}[2^x]$$

$$\Rightarrow \frac{d}{dx}\left[(e^{\ln 2})^x\right] \Rightarrow \frac{d}{dx}\left[e^{x\ln 2}\right]$$

$$= \underbrace{e^{\ln 2x}}_{2}\cdot\ln 2 = 2^x\ln 2$$

$$\int \ln x \, dx$$

$\Rightarrow x\ln x - \int 1 \, dx$

$= x \ln x - x$

	D	I
$+$	$\ln x$	1
$-$	$\frac{1}{x}$	x
$+$		
$-$		

$$\int \frac{2x^3}{x^4+1} \, dx$$

$\Rightarrow$ let $u = x^4+1$

$du = 4x^3 \, dx$

$dx = \frac{1}{4x^3} \, du \quad \Rightarrow \int \frac{2\cancel{x^3}}{u} \frac{1}{4\cancel{x^3}} \, du$

$= \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln|u| + c$

$= \frac{1}{2} \ln|x^4+1| + c$

$$\int \frac{x^2 - 3x + 2}{x+1} \, dx$$

$$\Rightarrow \begin{array}{r|l} & x - 4 \\ \hline x+1 & x^2 - 3x + 2 \\ & \underline{x^2 + x \quad\quad} \\ & -4x + 2 \\ & \underline{-4x - 4} \\ & 6 \end{array}$$

$\Rightarrow \int (x-4) + \frac{6}{x+1} \, dx$

$= \frac{1}{2}x^2 - 4x + 6\ln|x+1| + c$

$$\int \frac{1}{x(1+2\ln x)} \, dx$$

let $u = 1 + 2\ln x$

$du = 2\frac{1}{x} \, dx$

$dx = \frac{x}{2} \, du$

$\Rightarrow \int \frac{1}{\cancel{x} \cdot u} \frac{\cancel{x}}{2} \, du$

$= \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln|1 + 2\ln x| + c$

$$\int x e^{-x^2} \, dx$$

let $u = e^{-x^2}$

$du = e^{-x^2}(-2x)\,dx$

$\frac{1}{-2x}du = e^{-x^2}dx$

$\Rightarrow \int x \frac{1}{-2x}du \Rightarrow \int \frac{1}{-2}du = -\frac{1}{2}u + C$

$\Rightarrow -\frac{1}{2}e^{-x^2} + C$

$\int_{-1}^{1} \frac{1}{e^{2-3x}}dx$

$\Rightarrow \int_{-1}^{1} e^{-(2-3x)}dx$

$= \int_{-1}^{1} e^{3x-2}dx$

$\Rightarrow$ let $u = 3x-2$ $\Rightarrow \int_{-5}^{1} e^{u}\frac{1}{3}du$

$du = 3dx$

$= \frac{1}{3}\int_{-5}^{1} e^{u}du$

$= \frac{1}{3}[e^{u}]_{-5}^{1} = \frac{1}{3}[e - e^{-5}]$

$\int \log_4 x\,dx$

$\Rightarrow \int \frac{\ln x}{\ln 4}dx = \frac{1}{\ln 4}\int \ln x\,dx$

$= \frac{1}{\ln 4}[x\ln x - x] + C$

	D	I
+	$\ln x$	1
−	$\frac{1}{x}$	x
+		

$\Rightarrow x\ln x - \int 1\,dx$

$= x\ln x - x$

$\int \frac{e^{1/x^2}}{x^3}dx$

$\Rightarrow u = 1/x^2 = x^{-2}$ $\quad du = -2x^{-3}dx \quad dx = -\frac{1}{2}x^3du$

$\int \frac{e^u}{\cancel{x^3}}\cdot(-\frac{1}{2})\cancel{x^3}du = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u$

$= -\frac{1}{2}e^{x^{-2}} + C$

$\int \frac{e^{3x}+4e^{x}+5}{e^{x}}dx$

$\Rightarrow \int \frac{e^{3x}+4e^{x}+5}{e^{x}}dx = \int e^{2x} + 4 + 5e^{-x}dx$

$= \frac{1}{2}e^{2x} + 4x - 5e^{-x} + C$

$\int 2^x dx$

$\Rightarrow \int 2^x dx = \int e^{\ln 2^x}dx$

$\int e^{x\ln 2}\,d$

$$\Rightarrow \int 2^x\,dx = \int e^{\ln 2^x}\,dx$$
$$= \int e^{x\ln 2}\,dx$$

let $u = x\ln 2$

$du = \ln 2\,dx \qquad dx = \frac{1}{\ln 2}\,du$

$$\Rightarrow \int e^u \frac{1}{\ln 2}\,du$$
$$= \frac{1}{\ln 2}\int e^u\,du$$
$$= \frac{1}{\ln 2}e^u$$
$$= \frac{1}{\ln 2}e^{x\ln 2}$$
$$= \frac{1}{\ln 2}(e^{\ln 2})^x$$
$$= \frac{1}{\ln 2}2^x + C$$

$$\int \frac{1}{2^x}\,dx$$

$\Rightarrow \int 2^{-x}\,dx \Rightarrow$ let $y = 2^{-x}$

$\frac{d}{dx}[2^x] = 2^x\ln 2$ → 上面講過的

$\frac{d}{dx}[2^{-x}] = -2^{-x}\ln 2 \quad \Rightarrow \quad 2^{-x} = -\frac{\frac{d}{dx}[2^{-x}]}{\ln 2} = \frac{d}{dx}\left[-2^{-x}\cdot\frac{1}{\ln 2}\right]$

因為是常數，所以可以搬進去中

$$\Rightarrow \int 2^{-x}\,dx = \left[-2^{-x}\cdot\frac{1}{\ln 2}\right] + C$$

課本

P.187，第10題

⇒ 推導公式（給你 T_s，想找到 $T(t)$）

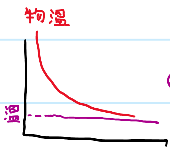
某個時間點下，物品的溫度

T：溫度對時間的關係

題目給的 → $\frac{dT}{dt} = k(T - T_s)$

（牛頓提出的）

初溫度



← 離室溫越遠下降越快

移項

$$\frac{1}{T-T_s}\,dT = k\,dt$$

同時積分

$$\ln|T-T_s| + c = kt + C$$

合併（因為不管c是多少）

$$\ln|T-T_s| = kt + c$$

同時變成 e^x

$$T - T_s = e^{kt+c} = e^{kt}\cdot e^c$$

也是一個常數，看作c

$$T - T_s = c\cdot e^{kt}$$

成 e^x

$$T - T_s = c \cdot e^{kt}$$ 也是一個常數，看作C

$$T = T_s + ce^{kt} = T(t)$$

if $T_s = 7°C$, $T(0) = 22°C$, $T(1) = 16°C$

(1) temperature of the soda after 1.5 hour?

$$22 = 7 + Ce^{kt}$$ t代入0

$$22 = 7 + C \Rightarrow C = 15$$

$$16 = 7 + 15e^{k \cdot 1} \Rightarrow 15e^k = 9$$

$$e^k = \frac{9}{15} = \frac{3}{5}$$

$$T(1.5) = 7 + 15\left(\frac{3}{5}\right)^{1.5} = 13.97$$

(2) How long does it take for the soda to cool to 10°C?

$$T(s) = 7 + 15\left(\frac{3}{5}\right)^s = 10$$

$$3 = 15\left(\frac{3}{5}\right)^s$$

$$\left(\frac{3}{5}\right)^s = \frac{1}{5} \Rightarrow \ln\frac{1}{5} = s\ln\left(\frac{3}{5}\right)$$

$$\Rightarrow s = 3.15 \text{ (hr)}$$

$$\int \frac{x^2-4}{x}\,dx$$

$$\Rightarrow \int x - \frac{4}{x}\,dx$$

$$= \frac{1}{2}x^2 - 4\ln|x| + C$$

$$\int \frac{(\ln x)^2}{x}\,dx$$

$\Rightarrow$ let $u = \ln x$

$$du = \frac{1}{x}dx$$

$$dx = x\,du$$

$$\Rightarrow \int \frac{u^2}{x} x\,du = \int u^2\,du = \frac{1}{3}u^3$$

$$= \frac{1}{3}(\ln x)^3 + C$$

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$$

$\Rightarrow$ let $u = e^x - e^{-x}$

$$du = (e^x + e^{-x})\,dx$$

$$dx = \frac{1}{e^x + e^{-x}}\,du$$

$du = (e^{\cdot} + e^{\cdot})\,dx$

$dx = \dfrac{1}{e^x + e^{-x}}\,du$

$\Rightarrow \displaystyle\int \frac{\cancel{e^x + e^{-x}}}{u} \cdot \frac{1}{\cancel{e^x + e^{-x}}}\,du$

$= \displaystyle\int \frac{1}{u}\,du$

$= \ln|u| = \ln|e^x - e^{-x}| + C$

$$\int \frac{x^3 - 3x^2 + 5}{x - 3}\,dx$$

$$\Rightarrow \begin{array}{r|l} & x^2 + 0 + 0 \\ \hline x-3 & x^3 - 3x^2 + 0 + 5 \\ & \underline{x^3 - 3x^2 \qquad\quad} \\ & \qquad\qquad\qquad 5 \end{array}$$

$\Rightarrow \displaystyle\int x^2 + \frac{5}{x-3}\,dx$

$= \frac{1}{3}x^3 + 5\ln|x-3| + C$