積分技巧-分部積分

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公式：$\int_a^b u(x)\,v'(x)\,dx = \Big[u(x)\,v(x)\Big]_a^b - \int_a^b u'(x)\,v(x)\,dx$

or $\int u\,dv = uv - \int v\,du$

$\int u\,dv = uv - \underline{\int v\,du}$ 有時做到這裡還需要再一次分部積分

⇒ 用表格法做會比較方便

表格法：

	D	I
+	u	dv
−	$du = t$	$v = ds$
+	dt	s
−	⋮ (微分下去)	⋮ (積分下去)

左邊記得要正負正…

$$\Rightarrow \int u\,dv = \underbrace{uv}_{\text{斜的}} - \underbrace{\int v\,du}_{\text{橫的(最後一列)}}$$
$$= uv - \left(\int t\,ds\right)$$
$$= uv - \left(ts - \int s\,dt\right)$$
$$= \underbrace{uv - ts}_{\text{斜的}} + \underbrace{\int s\,dt}_{\text{橫的 (最後一列)}}$$

$\int x^2 \sin(3x)\,dx$

微分(Differentiate)　積分(Integral)

	D	I
+	x^2	$\sin(3x)$
−	$2x$	$-\frac{1}{3}\cos(3x)$
+	2	$-\frac{1}{9}\sin(3x)$
−	0	$\frac{1}{27}\cos(3x)$

① 做到微分的變0

$$x^2\cdot\left(-\tfrac{1}{3}\cos(3x)\right) + \tfrac{2}{9}x\sin(3x) + \tfrac{2}{27}\cos(3x) + \int 0\cdot\tfrac{1}{27}\cos(3x)\,dx$$
$$= -\frac{x^2}{3}\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x)$$

$\int x^4 \ln x\,dx$

	D	I
+	$\ln x$	x^4
−	$\frac{1}{x}$	$\frac{1}{5}x^5$
+		
−		

② 做到可以積分

$$\ln x\cdot\tfrac{1}{5}x^5 - \int \tfrac{1}{x}\cdot\tfrac{1}{5}x^5\,dx$$
$$= \tfrac{1}{5}x^5\ln x - \tfrac{1}{5}\int x^4\,dx$$
$$= \tfrac{1}{5}x^5\ln x - \tfrac{1}{25}x^5 + c$$

$\int e^x \sin x\,dx$

	D	I
+	e^x	$\sin x$
−	e^x	$-\cos x$
+	e^x	$-\sin x$
−		

③ 做到和題目一樣的

移項

$$\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$$
$$\Rightarrow 2\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x$$
$$\int e^x\sin x\,dx = \frac{-1}{2}e^x\cos x + \frac{1}{2}e^x\sin x + C$$

$$\int \frac{\ln x}{x^3}\,dx$$

	D	I
+	$\ln x$	x^{-3}
−	$\frac{1}{x}$	$-\frac{1}{2}x^{-2}$
+		
−		

$$\Rightarrow -\frac{1}{2}x^{-2}\ln x + \int \frac{1}{x}\cdot\frac{1}{2}x^{-2}\,dx$$

$$= -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}\,dx$$

$$= -\frac{1}{2}x^{-2}\ln x + \frac{1}{-4}x^{-2} + C$$

$$43\int x\ln x\,dx$$

	D	I
+	$\ln x$	x
−	$\frac{1}{x}$	$\frac{1}{2}x^2$

$$\Rightarrow 43\left(\frac{1}{2}x^2\ln x - \int \frac{1}{2}x\right)$$

$$= 43\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) + C$$

$$\int x\cos 3x\,dx$$

	D	I
+	x	$\cos 3x$
−	1	$\frac{1}{3}\sin 3x$
+	0	$-\frac{1}{9}\cos 3x$

$$\Rightarrow \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$$

$$\int x\sqrt{x+19}\,dx$$

	D	I
+	x	$(x+19)^{\frac{1}{2}}$
−	1	$\frac{2}{3}(x+19)^{\frac{3}{2}}$
+	0	$\frac{4}{15}(x+19)^{\frac{5}{2}}$

$$\Rightarrow \frac{2}{3}x(x+19)^{\frac{3}{2}} - \frac{4}{15}(x+19)^{\frac{5}{2}} + C$$

$$\int t^2 e^{3t}\,dt$$

	D	I
+	t^2	e^{3t}
−	$2t$	$\frac{1}{3}e^{3t}$
+	2	$\frac{1}{9}e^{3t}$
−	0	$\frac{1}{27}e^{3t}$

$$\Rightarrow \frac{1}{3}t^2e^{3t} - \frac{2}{9}te^{3t} + \frac{2}{27}e^{3t} + C$$

$$\int_0^1 \arctan x\,dx$$

$\int_0^1 \arctan x \, dx$

$\arctan 1 = \frac{\pi}{4} = 45^\circ$

	D	I
+	$\arctan x$	1
−	$\frac{1}{1+x^2}$	x
+		

$\Rightarrow \left[x \arctan x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$

let $u = x^2+1$
$du = 2x\,dx$
$dx =$

$= \frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1$

$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

$\int \sinh^{-1} x \, dx$

☆ $\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{x^2+1}}$

	D	I
+	$\sinh^{-1} x$	1
−	$\frac{1}{\sqrt{x^2+1}}$	x

$\Rightarrow x\sinh^{-1} x - \underbrace{\int \frac{x}{\sqrt{x^2+1}}\, dx}$

let $u = x^2+1$
$du = 2x\,dx$
$dx = \frac{1}{2x}\,dx$

$= x\sinh^{-1} x - \int \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{1}{2\cancel{x}}\, dx$

$= x\sinh^{-1} x - \frac{1}{2}\int u^{-\frac{1}{2}}\, dx$

$= x\sinh^{-1} x - \frac{1}{2} \cdot 2\, u^{\frac{1}{2}}$

$= x\sinh^{-1} x - \sqrt{x^2+1} + C$

$\int \tanh^{-1} x \, dx$

☆ $\frac{d}{dx}[\tanh^{-1} x] = \frac{1}{1-x^2}$

	D	I
+	$\tanh^{-1} x$	1
−	$\frac{1}{1-x^2}$	x

$\Rightarrow x\tanh^{-1} x - \underbrace{\int \frac{x}{1-x^2}\, dx}$

let $1-x^2 = u$
$du = -2x\,dx$
$dx = -\frac{1}{2x}\,du$

$\Rightarrow x\tanh^{-1} x - \int \frac{x}{u} \cdot -\frac{1}{2x}\, du$

$= x\tanh^{-1} x + \frac{1}{2}\int \frac{1}{u}\, du$

$$= x\tanh^{-1}x + \frac{1}{2}\ln|u|$$

$$= x\tanh^{-1}x + \frac{1}{2}\ln|1-x^2| + C$$

$$\int x\sin x\cos x\,dx$$

	D	I
+	x	$\sin x\cos x$
−	1	$\frac{1}{2}\sin^2 x$

$$= \frac{1}{2}x\sin^2 x - \frac{1}{2}\int \sin^2 x\,dx \qquad \bigstar\ \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}x\sin^2 x - \frac{1}{2}\int \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{1}{2}x\sin^2 x - \frac{1}{2}\int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$$

$$= \frac{1}{2}x\sin^2 x - \frac{1}{2}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)$$

$$= \frac{1}{2}x\sin^2 x - \frac{1}{4}x + \frac{1}{8}\sin 2x + C$$

$$\int \left(\frac{\ln x}{x}\right)^2 dx$$

$$\Rightarrow \int \frac{(\ln x)^2}{x^2}\,dx = \int (\ln x)^2 \cdot x^{-2}\,dx$$

	D	I
+	$(\ln x)^2$	x^{-2}
−	$2\ln x \cdot \frac{1}{x}$	$-x^{-1}$

$$\Rightarrow (\ln x)^2 \cdot -\frac{1}{x} + 2\int \frac{\ln x}{x^2}\,dx$$

	D	I
+	$\ln x$	x^{-2}
−	$\frac{1}{x}$	$-x^{-1}$

$$\Rightarrow -\frac{(\ln x)^2}{x} - 2\left(\frac{\ln x}{x} + \int x^{-2}\,dx\right)$$

$$= -\frac{(\ln x)^2}{x} - \frac{2\ln x}{x} - \frac{2}{x} + C$$

$$\int x^5 e^{x^2}\,dx$$

	D	I
+	x^4	xe^{x^2}
−	$4x^3$	$\frac{1}{2}e^{x^2}$

$$\Rightarrow \frac{1}{2}x^4 e^{x^2} - 2\int x^3 e^{x^2}\,dx$$

$-\ 4x^3 \to \frac{1}{2}e^{x^2}$

D	I
$+\ x^2$	xe^{x^2}
$-\ 2x$	$\frac{1}{2}e^{x^2}$

$$\Rightarrow \frac{1}{2}x^4e^{x^2} - 2\left(\frac{1}{2}x^2e^{x^2} - \int xe^{x^2}\,dx\right)$$
$$= \frac{1}{2}x^4e^{x^2} - 2\left(\frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2}\right)$$
$$= \frac{1}{2}x^4e^{x^2} - x^2e^{x^2} + e^{x^2} + C$$

or

let $u = x^2$ $\quad du = 2x\,dx$ $\quad dx = \frac{1}{2x}du$ $\quad \Rightarrow \int x^5 e^u \cdot \frac{1}{2x}\,du$

$$\Rightarrow \frac{1}{2}\int x^4 e^u\,du$$
$$= \frac{1}{2}\int u^2 e^u\,du$$
$$= \frac{1}{2}\left(u^2e^u - 2ue^u + 2e^u\right)$$
$$= \frac{1}{2}\left(x^4e^{x^2} - 2x^2e^{x^2} + 2e^{x^2}\right) + C$$

D	I
$+\ u^2$	e^u
$-\ 2u$	e^u
$+\ 2$	e^u
$-\ 0$	e^u

$$\int x\,2^x\,dx$$

D	I
$+\ x$	2^x
$-\ 1$	$\frac{2^x}{\ln 2}$
$+\ 0$	$\frac{2^x}{(\ln 2)^2}$

$$\int 2^x\,dx$$
$$= \int (e^{\ln 2})^x\,dx$$
$$= \int e^{x\ln 2}\,dx = \int e^u \frac{1}{\ln 2}\,du = \frac{1}{\ln 2}e^u$$
$$= \frac{e^{x\ln 2}}{\ln 2}$$
$$= \frac{(e^{\ln 2})^x}{\ln 2}$$
$$= \frac{2^x}{\ln 2}$$

$\Rightarrow$ let $u = x\ln 2$
$du = \ln 2\,dx$
$dx = \frac{1}{\ln 2}du$

$$\Rightarrow \frac{x\,2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + C$$

$$\int x^3\sqrt{4-x^2}\,dx$$

D	I
$+\ x^2$	$x(4-x^2)^{\frac{1}{2}}$
$-\ 2x$	$-\frac{1}{3}(4-x^2)^{\frac{3}{2}}$

$$\Rightarrow -\frac{1}{3}x^2(4-x^2)^{\frac{3}{2}} + \frac{2}{3}\int x(4-x^2)^{\frac{3}{2}}\,dx$$
$$= -\frac{1}{3}x^2(4-x^2)^{\frac{3}{2}} - \frac{2}{15}(4-x^2)^{\frac{5}{2}} + C$$

$$\int e^{at}\sin t\,dt$$

	D	I
$+$	e^{at}	$\sin t$
$-$	ae^{at}	$-\cos t$
$+$	a^2e^{at}	$-\sin t$

$$\Rightarrow \int e^{at}\sin t\,dt = -e^{at}\cos t + ae^{at}\sin t - a^2\int e^{at}\sin t\,dt$$

$$(1+a^2)\int e^{at}\sin t\,dt = -e^{at}\cos t + ae^{at}\sin t$$

$$\int e^{at}\sin t\,dt = \frac{-e^{at}\cos t + ae^{at}\sin t}{1+a^2} + C$$

Ⓟ $$\int \ln^2(x^{20})\,dx$$

	D	I
$+$	$\ln^2(x^{20})$	1
$-$	$2\ln(x^{20})\cdot\frac{20x^{19}}{x^{20}}$	x

$$\Rightarrow x\ln^2(x^{20}) - 40\int \ln(x^{20})\,dx$$

	D	I
$+$	$\ln(x^{20})$	1
$-$	$\frac{20x^{19}}{x^{20}}$	x

$$\Rightarrow x\ln^2(x^{20}) - 40\left(x\ln(x^{20}) - \int 20\,dx\right)$$

$$= x\ln^2(x^{20}) - 40x\ln(x^{20}) + 800x + C$$

$$\int \sin(\ln x)\,dx$$

let $u=\ln x \Rightarrow x=e^u$

$du=\frac{1}{x}dx$

$dx = x\,du$

$$\Rightarrow \int e^u \sin u\,du$$

	D	I
$+$	$\sin u$	e^u
$-$	$\cos u$	e^u
$+$	$-\sin u$	e^u

$$\int e^u\sin u\,du = e^u\sin u - e^u\cos u - \int e^u\sin u\,du$$

$$2\int e^u\sin u\,du = e^u\sin u - e^u\cos u$$

$$\int e^u\sin u\,du = \frac{1}{2}\left(e^u\sin u - e^u\cos u\right)$$

$$= \frac{1}{2}\left(e^{\ln x}\sin(\ln x) - e^{\ln x}\cos(\ln x)\right)$$

$$= \frac{1}{2}\left(x\sin(\ln x) - x\cos(\ln x)\right)$$